

# Universality and tree structure of high energy QCD

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Using non-trivial mathematical properties of a class of nonlinear evolution equations, we obtain the universal terms in the asymptotic expansion in rapidity of the saturation scale and of the unintegrated gluon density from the Balitskiĭ-Kovchegov equation. These terms are independent of the initial conditions and of the details of the equation. The last subasymptotic terms are new results and complete the list of all possible universal contributions. Universality is interpreted in a general qualitative picture of high energy scattering, in which a scattering process corresponds to a tree structure probed by a given source.

## 1. Introduction

The energy dependence of hard hadronic cross sections is encoded in the evolution of the parton distributions with the rapidity  $Y$ . If the relevant scale  $Q$  is large enough, like in the case of deep-inelastic scattering, for which  $Q$  is the virtuality of the photon, the latter can be computed in perturbative QCD. The Balitskiĭ-Fadin-Kuraev-Lipatov (BFKL) equation [1] resums the leading Feynman diagrams, in the form of a linear integrodifferential evolution equation for the gluon density. Its solution exhibits an exponential growth with  $Y$ . At larger  $Y$ , maybe already attainable at present colliders [2], the BFKL equation breaks down because finite parton density or *saturation* effects become sizeable [3, 4]. It is replaced by nonlinear evolution equations which aim at describing such effects [5, 6, 7]. The appropriate equations depend on the physical observables and on the level of approximation of QCD. They could also include non-perturbative contributions [8]. Our aim is to show that minimal assumptions on the structure of saturation equations lead to quite general, i.e. “universal”, properties of the solutions. As a case study, we will consider the Balitskiĭ-Kovchegov (BK) equation [5, 7].

In the large  $N_c$  limit, at fixed QCD coupling  $\bar{\alpha} = \alpha_s N_c / \pi$ , and for a homogeneous nuclear target, the measured gluon density  $\mathcal{N}(k, Y)$  at transverse momentum  $k$  obeys the BK equation, which reads

$$\partial_Y \mathcal{N} = \bar{\alpha} \chi(-\partial_L) \mathcal{N} - \bar{\alpha} \mathcal{N}^2. \quad (1)$$

$\partial_L$  stands for the partial derivative with respect to  $\log k^2$  and the function  $\chi$  is the BFKL kernel  $\chi(\gamma) = 2\psi(1) - \psi(\gamma) - \psi(1 - \gamma)$ . The nonlinear term in the r.h.s. corresponds to the saturation term. For its relative simplicity,

the BK equation (1) is the prototype of saturation equations.

However, besides the mathematical challenge posed by the nonlinear character of saturation equations, there is up to now no fully consistent way to include saturation corrections in the perturbative expansion of QCD. Furthermore, initial conditions are required at some low rapidity  $Y_0$ , for all values of  $k$ : it is a priori not clear how dependent on them the solution of Eq. (1) is, in particular for small  $k \sim \Lambda_{\text{QCD}}$ , which lies beyond the known perturbative domain. Such sensitivity on the infrared has already been a major drawback to the BFKL equation [9]. One purpose of this Letter is to explain in what respect the solutions to the BK equation are independent of the details of both the nonlinear evolution equation and of the initial conditions. Let us call this property *universality*.

## 2. Universal properties of the solutions of the BK equation

We are going to use general methods for solving nonlinear evolution equations [10, 11]. The mathematical results that will be relevant for the BK equation require that three main conditions be fulfilled [11]. (i)  $\mathcal{N} = 0$  should be a linearly unstable fixed point, i.e. any small fluctuation of  $\mathcal{N}$  grows to infinity from the linearized evolution equation; (ii) the effect of the nonlinearity is to damp this growth when  $\mathcal{N} \sim \mathcal{O}(1)$ ; (iii) the initial condition must be “sufficiently” steep at large  $k$ . Properties (i) and (ii) stem from the parton model in general and are verified by the BK equation in particular. In the case of the BK equation, as noticed in Refs. [12, 13], the constraint (iii) means  $\mathcal{N}(k, Y_0) \ll 1/k^{2\gamma_c}$ , with  $\gamma_c = 0.6275 \dots$ . Color transparency of perturbative QCD implies  $\mathcal{N}(k, Y_0) \sim 1/k^2$  at large  $k$ , which fulfills this condition.

The class of equations defined by the latter constraints

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admit traveling wave solutions<sup>1</sup>, i.e. at large  $Y$ ,  $\mathcal{N}$  is a uniformly translating front:

$$\mathcal{N}(k, Y) \sim \mathcal{N}_\infty (\log k^2 - \log Q_s^2(Y)) . \quad (2)$$

The mathematical treatment of Eq. (1) consists in considering asymptotic expansions in  $Y$  of both the saturation scale (related to the translation of the traveling wave front with rapidity [12]) and of the gluon distribution  $\mathcal{N}$  (which is the front profile [13]). It leads to a hierarchical system of nested ordinary differential equations. At each order of the expansion, the integration constants are determined by a matching procedure [11]. The corresponding general equations can be straightforwardly translated to the case of the BK equation.

Let us first discuss the saturation scale  $Q_s(Y)$ . In terms of the parameters of the BK equation, the systematic expansion<sup>2</sup> of  $Q_s(Y)$  in  $Y$  reads

$$\begin{aligned} \log Q_s^2(Y) = & \bar{\alpha} \frac{\chi(\gamma_c)}{\gamma_c} Y - \frac{3}{2\gamma_c} \log Y \\ & - \frac{3}{\gamma_c^2} \sqrt{\frac{2\pi}{\bar{\alpha}\chi''(\gamma_c)}} \frac{1}{\sqrt{Y}} + \mathcal{O}(1/Y) . \end{aligned} \quad (3)$$

$\gamma_c$  is the solution of the implicit equation  $\chi(\gamma_c) = \gamma_c \chi'(\gamma_c)$ , see Ref. [3]. The successive orders contributing to  $\partial \log Q_s^2(Y)/\partial Y$  are displayed in Fig. 1.

A few comments are in order. The first term in Eq. (3) is the dominant evolution term [16]. The second term [17] reflects the delay in the formation of the front [13]. The third term is a new result for the solution of the BK equation. These three terms are universal. They are the only possible universal terms in this asymptotic expansion. Indeed, one sees that by performing a shift in the rapidity  $Y \rightarrow Y + Y_0$  in Eq. (3), that amounts to modifying the initial conditions, one gets additional  $1/Y$  terms, while the lowest order terms are not affected by the shift.

We now turn to the discussion of the properties of the front. The whole method to get solutions relies on the matching between two different expansions of the wave front above the saturation scale: the so-called “front interior” and “leading edge” expansions [11]. In the “front interior”, the relevant small parameter is  $z = \log(k^2/Q_s^2(Y))/\sqrt{2\bar{\alpha}\chi''(\gamma_c)Y} \ll 1$ , i.e. one stands at transverse momenta  $k$  near the saturation scale  $Q_s(Y)$ .

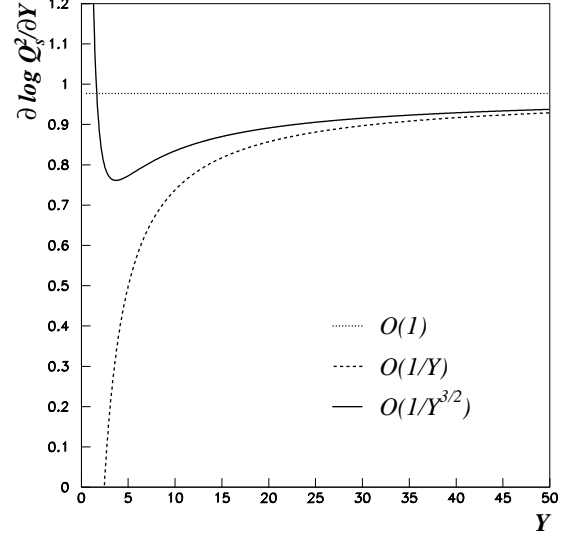


FIG. 1: The effective dependence  $\partial \log Q_s^2(Y)/\partial Y$  of the saturation scale upon the rapidity  $Y$ .  $\bar{\alpha}$  was set to 0.2. The truncations of the asymptotic expansion Eq. (3) to order 1,  $1/Y$  and  $1/Y^{3/2}$  are shown.

$\mathcal{N}$  is expanded about its asymptotic shape  $\mathcal{N}_\infty$ , see Eq. (2). In that region,  $\mathcal{N} \sim 1$  and the full nonlinear equation must be considered. In the “leading edge” defined by  $z \sim 1$ , the wave front is expanded in powers of  $1/\sqrt{Y}$ : it is the transition region, where the front forms. Going from the front interior to the leading edge, the asymptotic front profile  $\mathcal{N}_\infty$  crosses over to [11]

$$\begin{aligned} \mathcal{N}(k, Y) = & C_1 \left( \frac{k^2}{Q_s^2(Y)} \right)^{-\gamma_c} e^{-z^2} \times \left\{ \gamma_c \log[k^2/Q_s^2(Y)] \right. \\ & + C_2 + \left( 3 - 2C_2 + \frac{\gamma_c \chi^{(3)}(\gamma_c)}{\chi''(\gamma_c)} \right) z^2 \\ & - \left( \frac{2}{3} \frac{\gamma_c \chi^{(3)}(\gamma_c)}{\chi''(\gamma_c)} + \frac{1}{3} {}_2F_2 \left[ 1, 1; \frac{5}{2}, 3; z^2 \right] \right) z^4 \\ & \left. + 6\sqrt{\pi} \left( 1 - {}_1F_1 \left[ -\frac{1}{2}, \frac{3}{2}; z^2 \right] \right) z + \mathcal{O}(1/\sqrt{Y}) \right\} \end{aligned} \quad (4)$$

where  $C_1$  and  $C_2$  are two integration constants and  ${}_1F_1$ ,  ${}_2F_2$  are hypergeometric functions.

The first term can be identified to  $\mathcal{N}_\infty \times e^{-z^2}$  near the front, and reproduces the known results [13, 16, 17]:  $\mathcal{N}_\infty$  induces so-called geometric scaling [18], while  $e^{-z^2}$  describes the diffusive formation of the front, and induces a specific pattern of geometric scaling violations. All other terms appearing in the expansion (4) are new terms that represent the  $\mathcal{O}(1/\sqrt{Y})$  corrections to  $\mathcal{N}_\infty \times e^{-z^2}$ . Note that they depend also on the third derivative  $\chi^{(3)}(\gamma_c)$ , at variance with the new term of

<sup>1</sup> Rigorous mathematical results are available [14] for the Fisher and Kolmogorov-Petrovsky-Piscounov [15] equation (related to the BK equation in the diffusive approximation [12]), and have been extended to more general cases [11].

<sup>2</sup> Note that the scale of  $Q_s(Y)$  is included in the definition. Indeed, the saturation scale  $Q_s(Y)$  can be defined, e.g., in such a way that  $\mathcal{N}(Q_s(Y), Y) = \mathcal{N}_0$ , where  $\mathcal{N}_0$  is a given constant.  $Q_s(Y)$  is the position of the traveling wave front. All universal terms are independent of  $\mathcal{N}_0$ .

order  $1/\sqrt{Y}$  in the expression for the saturation scale (3).

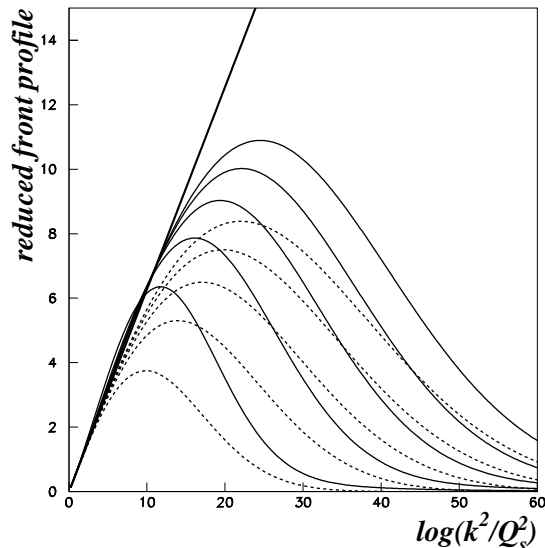


FIG. 2: The reduced front profile  $\mathcal{N}(k, Y) \times (k^2/Q_s^2(Y))^{\gamma_c}$  in the leading edge. The curves are for  $Y = 10$  (lower curves) to  $Y = 50$  (upper curves).  $\bar{\alpha} = 0.2$  and the constants in Eq. (4) were set to  $C_1 = 1$ ,  $C_2 = 0$ . The leading edge expansion (4) is truncated at level  $\mathcal{O}(\sqrt{Y})$  (dashed lines) and  $\mathcal{O}(1)$  (solid lines). The reduced asymptotic front, i.e.  $\mathcal{N}_\infty(k, Y) \times (k^2/Q_s^2(Y))^{\gamma_c}$  (see Eq. (2)), is the straight line.

### 3. Physical interpretation of universality

Let us discuss the physical picture underlying the universality properties of a general scattering process in high energy QCD, beyond the specific example of the BK equation. The linear kernels of the saturation equations correspond to a developing tree of partons. The tree structure is generated by successive branching of partons when the rapidity  $Y$  increases, which stems from the non-abelian character of QCD. For example, the BFKL kernel is known to correspond to a specific tree of gluons or, equivalently at large  $N_c$ , of cascading dipoles [19].

In the course of the parton branching, the partons get more densely packed together, and thus the color field gets stronger. The strength of the field can be qualitatively characterized by the average transverse distance  $d(Y)$  between neighboring partons.

The scattering process corresponds to the interaction of a source with the partons. As long as  $d(Y)$  is larger than the resolution  $1/Q$  of the probe, the field seen by the probe is weak and thus the interaction merely “counts” the partons. This is the dilute regime, in which the evolution of the observable follows the exponential growth of the parton density with  $Y$  given by the BFKL equation. When  $d(Y)$  reaches  $1/Q$ , the color field seen by the

probe becomes strong, and the probe can interact with several partons simultaneously, see Fig. 3. This is the saturation transition, and  $d(Y)$  plays the role of the saturation size  $1/Q_s(Y)$ .<sup>3</sup> As  $d(Y)$  gets even smaller, the partons inside the tree may interact and recombine. All these finite density or saturation effects generate nonlinear damping terms in the evolution equations that slow down the evolution. Depending on the physical situation, the evolution of the interaction amplitude in this regime is e.g. described by the BK equation, or by the Balitskiĭ [5] and JIMWLK [6] equations which include more saturation effects.

The structure described here complies with the three abovementioned conditions (i), (ii), (iii) for an evolution equation to admit traveling wave solutions. The previous discussion is general and applies to various modifications of the saturation equations. Therefore, we expect the universal results presented here to be valid in general, up to the replacement of the relevant parameters related to the basic tree and which characterize the linear evolution kernel.

### 4. Conclusion and discussion

We have pointed out that the structure of the QCD parton model is related to general cascading processes possessing specific mathematical properties. We have obtained all universal terms including new ones in the asymptotic expansion of the saturation scale and of the unintegrated gluon distribution, see Eqs. (3,4). Universality means independence of the solution with respect to the initial conditions after a long enough rapidity evolution and dependence of the solution on a few relevant parameters obtained from the kernel of the *linear* evolution:  $\gamma_c$ ,  $\chi(\gamma_c)$ ,  $\chi''(\gamma_c)$  for the saturation scale, plus  $\chi^{(3)}(\gamma_c)$  for the front. The results have been obtained from the BK equation, but are straightforwardly applicable to a whole class of saturation equations. The saturation scale and the gluon density above the saturation scale keep exactly the same functional form for the universal terms.

Finally, it is interesting to look for the concrete imprints of saturation on the universal parts of the solution, Eqs. (3,4). They appear in two places: the coefficient of the second term of the expansion of the saturation scale ( $\frac{3}{2}$  instead of  $\frac{1}{2}$  for BFKL, see Eq. (3)), and the logarithmic factor in  $\mathcal{N}_\infty$  in the expression for the asymptotic front (Eq. (4)), that is responsible for the developing linear shape of the reduced front profile in the leftmost part of the leading edge, see Fig. 2. This factor appears to be a general dynamical property of absorbing walls, near

<sup>3</sup> This phenomenon is conventionally interpreted as the onset of the black disk due to the filling of the phase space by partons of size  $1/Q$  [3].

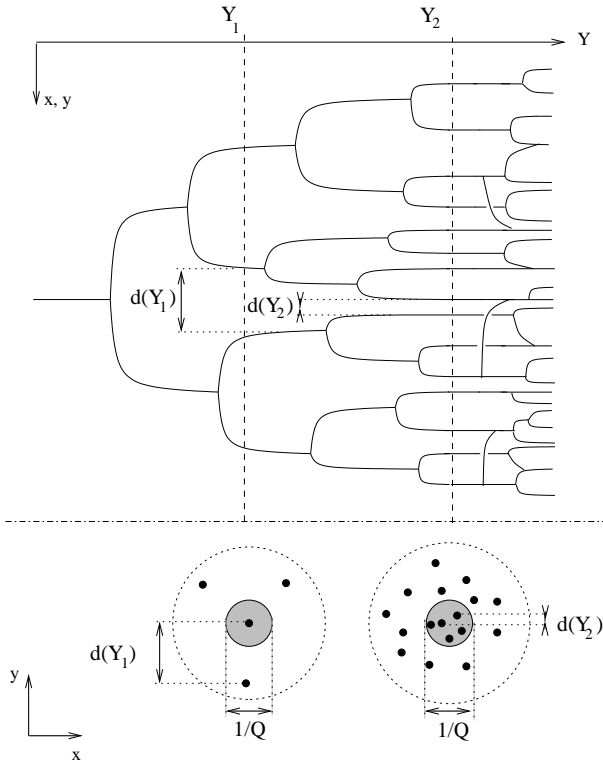


FIG. 3: *The tree structure.* The parton branching is represented along the rapidity axis (upper part) and in the transverse coordinate space at two different rapidities (lower part). The probe is represented by a shaded disk of size  $1/Q$ . At rapidity  $Y_1$ , the probe counts the partons (linear regime), at  $Y_2$ , the probe counts groups of partons (transition to saturation). This effect, and also other effects such as recombination of partons inside the tree (r.h.s. of the plot), generate nonlinear damping terms in the evolution equations.

which a linear gradient develops at long times [11]. The nonlinearities present in the saturation equations act effectively as an absorbing wall in the results of Eqs. (3,4). This a posteriori justifies the treatment of saturation proposed in Ref. [17].

Numerical solutions of the BK [20, 21] and JIMWLK [21] equations are available. The traveling wave behavior of the solution was seen, and the consistency with the first two universal terms in Eq. (3) was checked. It would be interesting to include the newly found term in the comparison between the numerical and analytical solutions. However, the expansion (3,4) is an asymptotic series, which is only slowly converging, see Figs. 1,2.

The question of the phenomenology of saturation is still open. For realistic phenomenology, one should go beyond leading order BK (see e.g. [22]) since, as well-known, the saturation scale predicted by the leading order BK equation has a too steep rapidity evolution.

In a forthcoming publication [23], we take into account some of these subleading effects by extending the discussion of universality to the case of the BK equation

with running coupling. The latter falls in a different class of nonlinear equations, but it can be investigated through the same general methods, showing the power of the mathematical approach.

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